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An Algebraic View on Finite Rank in 2D-SSA

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Abstract

The 2D-SSA method provides a decomposition of a 2D-array (a function of two variables, e.g. digital image) into a sum of identifiable components. For the decomposition to be proper, these components should be close to 2D-arrays of finite rank. This paper is devoted to study of arrays of finite rank by means of polynomial ideals generated by arrays. The 2D-arrays are considered as functionals of polynomials. A general form of arrays of finite rank is obtained. The structure of finite-rank arrays and their trajectory spaces is investigated.

1 Introduction

The 2D-SSA method [5] is the two-dimensional extension of the well-known Singular Spectrum Analysis [2]. 2D-SSA deals with a 2D-array $\mathrm{F} = \mathrm{F}^{(N_x,N_y)} = (f_{i,j})_{i,j=0}^{N_x-1,N_y-1}$ and is aimed to decompose the 2D-array into a sum of components of different structure. The method has two parameters (L_x, L_y) called *window sizes*, $L_x \geq 1, L_y \geq 1, L_x L_y > 1$ and $L_x \leq N_x, L_y \leq N_y, L_x L_y < N_x N_y$. 2D-SSA considers $L_x \times L_y$ submatrices $\mathrm{F}_{k,l}^{(L_x,L_y)} \stackrel{\mathrm{def}}{=} (f_{i+k,j+l})_{i=0,j=0}^{L_x-1,L_y-1}$ and studies properties of the *trajectory space*

$$\mathcal{L}^{(L_x,L_y)}(\mathrm{F}) = \mathrm{span}(\{\mathrm{F}_{k,l}^{(L_x,L_y)}\}_{k,l=0}^{N_x-L_x,N_y-L_y}).$$

The dimension of $\mathcal{L}^{(L_x,L_y)}(\mathrm{F})$, referred to as *2D-SSA rank of* F, plays an important role in the theory of the 2D-SSA method.

In this paper, we consider an infinite complex-valued 2D-array $\mathcal{F} = (f_{i,j})_{i,j=0}^{+\infty}$ containing F as its submatrix. In the same manner, we introduce the *trajectory space of the infinite array* $\mathcal{F}$ (for $L_x \geq 1, L_y \geq 1, L_x L_y > 1$)

$$\mathcal{L}^{(L_x,L_y)}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \mathrm{span}(\{\mathrm{F}_{k,l}^{(L_x,L_y)}\}_{k,l=0}^{+\infty}),$$

which evidently contains $\mathcal{L}^{(L_x,L_y)}(\mathrm{F})$ as a subspace. If there exist d, L_{x0}, L_{y0} such that $\mathrm{rank}_{(L_x,L_y)}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = d$ for any $L_x \geq L_{x0}$ and $L_y \geq L_{y0}$, then $\mathcal{F}$ is said to be an *array of finite 2D-SSA rank*. We will show that arrays of this kind satisfy $\mathcal{L}^{(L_x,L_y)}(\mathrm{F}) = \mathcal{L}^{(L_x,L_y)}(\mathcal{F})$ if N_x and N_y are large enough.

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It is convenient to study properties of the trajectory space with the help of

$$\mathcal{L}(\mathcal{F}) \stackrel{\text{def}}{=} \operatorname{span}(\{\mathcal{F}_{k,l}\}_{k,l=0}^{+\infty}),$$

where $\mathcal{F}_{k,l}$ is the infinite array with entries $(\mathcal{F}_{k,l})_{i,j} = (\mathcal{F})_{i+k,j+l}$, called the (k,l)-*shift* of $\mathcal{F}$. The (k,l)-shifts, in their turn, can be studied by means of algebra of polynomials and polynomial ideals. It happens that this technique is quite appropriate for the 2D case, where the linear algebra approach appears to be insufficient, in contrast to the 1D case of time series of finite rank [2].

An infinite array is said to be an *array of finite rank* if $\operatorname{rank}\mathcal{F} \stackrel{\text{def}}{=} \dim\mathcal{L}(\mathcal{F}) < +\infty$. We will show that the 2D-array $\mathcal{F}$ is of finite rank iff (if and only if) it is of finite 2D-SSA rank. Note that an array of finite rank d has representation

$$f_{i+k,j+l} = \sum_{m=1}^{d} a_{i,j}^{(m)} b_{k,l}^{(m)}, \qquad (1)$$

where $\mathrm{A}^{(m)} = (a_{i,j}^{(m)})_{i,j=0}^{L_x-1,L_y-1}$, $m \in \{1,\ldots,d\}$, form the basis of $\mathcal{L}^{(L_x,L_y)}(\mathcal{F})$ and $b_{i,j}^{(m)}$ are some coefficients. Therefore, as a matter of fact, we treat arrays of type (1) when studying arrays of finite (2D-SSA) rank.

In Section 2 we introduce basic concepts of algebra of polynomials and polynomial ideals and establish a link between them and (k,l)-shifts of an infinite array. Then we study properties of infinite arrays of finite rank. Results of the section include a general form of arrays of finite rank. Section 3 contains properties of the trajectory space $\mathcal{L}^{(L_x,L_y)}(\mathcal{F})$ of an infinite array. Results of Section 3 state the equivalence of notions of finite rank and finite 2D-SSA rank.

2 Infinite arrays

2.1 Functionals of polynomials. Linear recurrent relations

Let V^* stand for the *dual space* (the space of all linear functionals $\ell : V \to \mathbb{C}$, see [3]) of a vector space V over $\mathbb{C}$. Let $\mathbb{P} = \mathbb{C}[x,y]$ denote the vector space of all polynomials in two variables. An infinite array $\mathcal{G} = (g_{i,j})_{i,j=0}^{+\infty}$ defines $\ell^{(\mathcal{G})} \in \mathbb{P}^*$ as follows. For $p(x,y) = \sum_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)} x^\rho y^\tau \in \mathbb{P}$, where $\#\{(\rho,\tau) : a_{(\rho,\tau)} \neq 0\} < +\infty$,

$$\ell^{(\mathcal{G})}(p) \stackrel{\text{def}}{=} \sum_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)} g_{\rho,\tau}. \qquad (2)$$

Let us denote $\ell_{k,l}^{(\mathcal{F})} \stackrel{\text{def}}{=} \ell^{(\mathcal{F}_{k,l})}$ and consider $\mathcal{D}(\mathcal{F}) \stackrel{\text{def}}{=} \operatorname{span}(\{\ell_{k,l}^{(\mathcal{F})}\}_{k,l=0}^{+\infty}) \in \mathbb{P}^*$. This space of functionals is isomorphic to $\mathcal{L}(\mathcal{F})$ (we write $\mathcal{D}(\mathcal{F}) \cong \mathcal{L}(\mathcal{F})$).

Definition 1. Let V be a vector space over $\mathbb{C}$. The *zero set* of a space of functions $S \subseteq (V \to \mathbb{C})$ is, by definition,

$$Z[S] \stackrel{\text{def}}{=} \{z \in V : \ f(z) = 0 \quad \forall f \in S\}.$$

Lemma 1. *The polynomial* $\sum\limits_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)} x^\rho y^\tau$ *belongs to* $Z[\mathcal{D}(\mathcal{F})]$ iff

$$\sum_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)} f_{k+\rho,l+\tau} = 0 \quad \text{for any } k, l \in \mathbb{N}_0, \tag{3}$$

or, that is equivalent, $\sum\limits_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)}\, \ell_{\rho,\tau}^{(\mathcal{F})} \equiv 0.$

In other words, $Z[\mathcal{D}(\mathcal{F})]$ consists of shift-invariant linear relations (3) between entries of $\mathcal{F}$. They are analogues of *linear recurrent formulae* in the 1D case (see [2]). Let us review some important properties of zero sets.

Definition 2. The *annihilator* of $Q \subset \mathbb{P}$ is defined by

$$A[Q] \stackrel{\text{def}}{=} \{\ell \in \mathbb{P}^* : \ \ell(p) = 0 \quad \forall p \in Q\}.$$

Proposition 1 ([1, Lemma 1.1]). $Z[A[Q]] = Q$ *for any subspace* Q *of* $\mathbb{P}$.

Remark 1. It is easy to see that $\mathcal{D} \subseteq A[Z[\mathcal{D}]]$ for any $\mathcal{D} \subset \mathbb{P}^*$.

Proposition 2 ([1, Cor. 1.7]). $\mathcal{D} = A[Z[\mathcal{D}]]$ *if the subspace* $\mathcal{D}$ *of* $\mathbb{P}^*$ *is finite-dimensional.*

2.2 Ideals. Closed spaces of functionals

Definition 3. A set of polynomials $\mathcal{I} \subset \mathbb{P}$ is a *polynomial ideal* if $p + sq \in \mathcal{I}$ for any $p, q \in \mathcal{I}$, $s \in \mathbb{P}$.

Definition 4. The *quotient ring* $\mathcal{R}[\mathcal{I}] = \mathbb{P}/\mathcal{I}$ of an ideal $\mathcal{I}$ is, by definition, the space of equivalence classes modulo $\mathcal{I}$:

$$\mathcal{R}[\mathcal{I}] \stackrel{\text{def}}{=} \{[p]_{\mathcal{I}} : \ p \in \mathbb{P}\}, \text{ where } [p]_{\mathcal{I}} \stackrel{\text{def}}{=} \{q \in \mathbb{P} : \ q - p \in \mathcal{I}\},$$

with multiplication and addition operations induced from $\mathbb{P}$ to $\mathcal{R}[\mathcal{I}]$.

Proposition 3. *The annihilator of an ideal* $\mathcal{I} \subseteq \mathbb{P}$ *is isomorphic to* $(\mathcal{R}[\mathcal{I}])^*$.

The proof is obvious since $\ell \in \mathcal{A}[\mathcal{I}]$ iff $\ell(p_1) = \ell(p_2)$ for any $p_1 \in \mathbb{P}$, $p_2 \in [p_1]_{\mathcal{I}}$. Hence, we can think of ℓ as of a function $\mathcal{R}[\mathcal{I}] \to \mathbb{C}$. For more details see [3, §2.3].

Definition 5. A vector space $\mathcal{D} \subset \mathbb{P}^*$ is called *closed* if

$$\forall q \in \mathbb{P} \quad \ell \in \mathcal{D} \Rightarrow (\ell \cdot q) \in \mathcal{D}, \quad \text{where } (\ell \cdot q)(p) \stackrel{\text{def}}{=} \ell(qp).$$

Proposition 4 ([3, §2.3.2]). *The annihilator of an ideal* $\mathcal{I} \subseteq \mathbb{P}$ *is closed.*

Proposition 5 ([3, Th. 2.21]). *For any closed space* $\mathcal{D} \subset \mathbb{P}^*$ *the zero set* $\mathcal{I}[\mathcal{D}] \stackrel{\text{def}}{=} Z[\mathcal{D}]$ *is a polynomial ideal.*

Let $p(x,y) = \sum\limits_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)} x^\rho y^\tau \in \mathbb{P}$. Then

$$(\ell_{k,l}^{(\mathcal{F})} \cdot x^\alpha y^\beta)\,(p) = \sum_{\rho,\tau=0}^{+\infty} a_{(\rho,\tau)} (\mathcal{F}_{k,l})_{\rho+\alpha,\tau+\beta} = \ell_{k+\alpha,l+\beta}^{(\mathcal{F})}\,(p)\,, \tag{4}$$

which allows us to prove the following assertion.

Proposition 6. *A vector space $\mathcal{D}(\mathcal{F})$ is closed.*

Thus the set of linear relations (3) has the structure of an ideal and can be studied by polynomial methods. For brevity we denote $\mathcal{I}(\mathcal{F}) \overset{\text{def}}{=} \mathcal{I}[\mathcal{D}(\mathcal{F})]$.

2.3 Zero-dimensional ideals and arrays of finite rank

Polynomials can be treated as functions $\mathbb{C}^2 \to \mathbb{C}$, therefore we may define the zero set $Z[\mathcal{I}] \subseteq \mathbb{C}^2$ of a polynomial ideal $\mathcal{I}$ (see Definition 1).

Definition 6. A polynomial ideal $\mathcal{I}$ is called *zero-dimensional* if its zero set is discrete, i.e. $Z[\mathcal{I}] = \{(\lambda_1,\mu_1),\ldots,(\lambda_n,\mu_n)\}$.

Theorem 1 ([4, Th. 3.1, 3.6]). *$\mathcal{I}$ is zero-dimensional iff* $\dim \mathcal{R}[\mathcal{I}] < +\infty$.

Applying Remark 1, Proposition 2 and Proposition 3 we obtain the following.

Corollary 1. *For a closed subspace $\mathcal{D}$, $\mathcal{I}[\mathcal{D}]$ is zero-dimensional iff* $\dim \mathcal{D} < +\infty$.

If $\dim \mathcal{D} < +\infty$ then $\mathcal{D} = \mathcal{A}[\mathcal{I}[\mathcal{D}]]$. Therefore $\mathcal{L}(\mathcal{F})$ is isomorphic to the annihilator of $\mathcal{I}(\mathcal{F})$ for an array of finite rank.

Definition 7. The *differential functional* $\partial_{(\alpha,\beta)}[\lambda,\mu] \in \mathbb{P}^*$ with $(\alpha,\beta) \in \mathbb{N}_0^2$ and $(\lambda,\mu) \in \mathbb{C}^2$ is defined by

$$\partial_{(\alpha,\beta)}[\lambda,\mu](p) \overset{\text{def}}{=} \frac{1}{\alpha!\beta!}\left(\frac{\partial^{\alpha+\beta} p}{\partial x^\alpha \partial y^\beta}\right)(\lambda,\mu).$$

Theorem 2 ([1, Th. 2.8]). *Let $Z[\mathcal{I}] = \{(\lambda_1,\mu_1),\ldots,(\lambda_n,\mu_n)\}$. Then*

$$A[\mathcal{I}] = \mathcal{D}_1 \oplus \ldots \oplus \mathcal{D}_n, \tag{5}$$

where D_k is a finite-dimensional closed subspace of $\operatorname{span}(\{\partial_{(\alpha,\beta)}[\lambda_k,\mu_k]\}_{(\alpha,\beta)\in\mathbb{N}_0^2})$.

Theorem 2 and the relation $f_{i,j} = \ell^{(\mathcal{F})}(x^i y^j)$ allow us to obtain the following general form of arrays of finite rank.

Proposition 7. *An infinite array $\mathcal{F}$ of finite rank has the form*

$$f_{i,j} = \sum_{k=1}^{n} q_k(i,j)\lambda_k^i \mu_k^j,$$

where $(\lambda_k,\mu_k) \in Z[\mathcal{I}(\mathcal{F})]$ and q_k are polynomials.

Applying Proposition 7 to real-valued arrays of finite rank gives

$$f_{i,j} = \sum_{k=1}^{h} p_k(i,j)\rho_k^i \tau_k^j \cos(\omega_k i + \alpha_k)\cos(\theta_k j + \beta_k),$$

where $\rho_k, \tau_k, \omega_k, \theta_k, \alpha_k, \beta_k \in \mathbb{R}$ and p_k are real polynomials.

3 Properties of trajectory spaces

3.1 Normal sets. Generators of ideal

For a set $\mathcal{B} \subset \mathbb{N}_0^2$, let $\mathcal{B} + (k,l) \stackrel{\text{def}}{=} \{(\alpha,\beta) \in \mathbb{N}_0^2 : (\alpha - k, \beta - l) \in \mathcal{B}\}$.

Definition 8. A set $\mathcal{A} \subset \mathbb{N}_0^2$, $\mathcal{A} \neq \varnothing$, is called a *normal set* of an ideal $\mathcal{I}$, if $(\mathcal{A} + (-1,0)) \cup (\mathcal{A} + (0,-1)) \subset A$ and $\left\{\left[x^\alpha y^\beta\right]\right\}_{(\alpha,\beta)\in\mathcal{A}}$ is a basis of $\mathcal{R}[\mathcal{I}]$.

For every ideal there exists a normal set (in most cases it is not unique). Let us consider a zero-dimensional ideal $\mathcal{I}$ and fix its normal set $\mathcal{A}$.

Lemma 2. *For any $(\alpha,\beta) \in \mathbb{N}_0^2 \setminus \mathcal{A}$ there exists unique polynomial*

$$p_{(\alpha,\beta)}(x,y) \stackrel{\text{def}}{=} x^\alpha y^\beta - \sum_{(\rho,\tau)\in\mathcal{A}} a_{(\alpha,\beta),(\rho,\tau)} x^\rho y^\tau \in \mathcal{I}.$$

Definition 9. A *generated by $Q \subset \mathbb{P}$ ideal* is, by definition, the set of finite polynomial combinations $\langle Q \rangle \stackrel{\text{def}}{=} \{g_1 h_1 + \ldots + g_m h_m : \ g_i \in Q,\ h_i \in \mathbb{P}\}$.

Proposition 8 ([3, Prop. 2.30]). *The ideal $\mathcal{I}$ is generated by $\left\{p_{(\alpha,\beta)}\right\}_{(\alpha,\beta)\in\delta(\mathcal{A})}$, where $\delta(\mathcal{A}) \stackrel{\text{def}}{=} \big((\mathcal{A} + (1,0)) \cup (\mathcal{A} + (0,1))\big) \setminus \mathcal{A}$.*

3.2 From ideals and functionals to trajectory spaces

Let $\mathcal{A}$ be a normal set of $\mathcal{I}(\mathcal{F})$. By Lemma 1 and Lemma 2 we obtain the following lemma.

Lemma 3. *The set $\{\ell_{k,l}^{(\mathcal{F})}\}_{(k,l)\in\mathcal{A}}$ is a basis of $\mathcal{D}(\mathcal{F})$.*

Lemma 3 implies that $\{\mathcal{F}_{k,l}\}_{(k,l)\in\mathcal{A}}$ is a basis of $\mathcal{L}(\mathcal{F})$. Let us fix some window sizes $(L_x, L_y) \in \mathbb{N}^2$ and deduce an analogous property for the trajectory space.

Definition 10. *The orthogonal complement of* $\mathcal{L}^{(L_x,L_y)}(\mathcal{F})$ is, by definition,

$$(\mathcal{L}^{(L_x,L_y)})_\perp \stackrel{\text{def}}{=} \{(a_{k,l})_{k,l=0}^{L_x-1,L_y-1} : \ \forall i,j \geq 0 \quad \textstyle\sum_{k,l} a_{k,l} f_{i+k,j+l} = 0\}.$$

Immediately, we get

$$(\mathcal{L}^{(L_x,L_y)})_\perp = \{(a_{k,l})_{k,l=0}^{L_x-1,L_y-1} : \textstyle\sum_{k,l} a_{k,l}\ell_{k,l}^{(\mathcal{F})} \equiv 0\}, \tag{6}$$

and the following proposition is evident.

Proposition 9. $\dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = \dim \operatorname{span}(\{\ell_{k,l}^{(\mathcal{F})}\}_{k,l=0}^{L_x-1,L_y-1})$.

Due to Lemma 3 and Proposition 9, we come to the equivalence of notions of finite rank and finite 2D-SSA rank.

Proposition 10. *$\mathcal{F}$ is of rank $d < +\infty$ iff there exist L_{x0}, L_{y0} such that*

$$\forall L_x \geq L_{x0},\ L_y \geq L_{y0} \quad \dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = d. \tag{7}$$

Having normal set $\mathcal{A}$, one can take in (7) $L_{x0} = \mathrm{B}_x(\mathcal{A}) \stackrel{\mathrm{def}}{=} \min\{\alpha : \mathcal{A} + (-\alpha, 0) = \varnothing\}$ and $L_{y0} = \mathrm{B}_y(\mathcal{A}) \stackrel{\mathrm{def}}{=} \min\{\beta : \mathcal{A} + (0, -\beta) = \varnothing\}$.

Proposition 11. *For $L_x > \mathrm{B}_x(\mathcal{A})$, $L_y > \mathrm{B}_y(\mathcal{A})$ the ideal $\mathcal{I}(\mathcal{F})$ is generated by*

$$Q_{\perp}^{(L_x,L_y)} \stackrel{\mathrm{def}}{=} \{\textstyle\sum_{k,l} a_{k,l} x^k y^l \in \mathbb{P} : (a_{k,l})_{k,l=0}^{L_x-1,L_y-1} \in (\mathcal{L}^{(L_x,L_y)})_{\perp}\}.$$

Proof. By (6) and Lemma 1, $Q_{\perp}^{(L_x,L_y)} \subset \mathcal{I}$. Obviously, $\{p_{(\alpha,\beta)}\}_{(\alpha,\beta)\in\delta(\mathcal{A})} \subset Q_{\perp}^{(L_x,L_y)}$. Therefore, by Proposition 8, $\mathcal{I} = \langle Q_{\perp}^{(L_x,L_y)} \rangle$. □

Proposition 12. *For $\mathcal{F}$, L_x, L_y such that $\dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = \operatorname{rank} \mathcal{F}$ and a normal set $\mathcal{A}$, the submatrices $\{\mathrm{F}_{k,l}^{(L_x,L_y)}\}_{(k,l)\in\mathcal{A}}$ form a basis of $\mathcal{L}^{(L_x,L_y)}(\mathcal{F})$.*

Proposition 12 means that a finite-size submatrix of an finite-rank infinite array inherits structure of this infinite array. Moreover, Proposition 11 implies that the entries of the infinite array are uniquely defined by its finite-size submatrix.

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